

BIFURCATION FROM INFINITY FOR AN ASYMPTOTICALLY LINEAR SCHRÖDINGER EQUATION

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ABSTRACT. We consider the asymptotically linear Schrödinger equation (1.1) and show that if λ_0 is an isolated eigenvalue for the linearization at infinity, then under some additional conditions there exists a sequence (u_n, λ_n) of solutions such that $\|u_n\| \rightarrow \infty$ and $\lambda_n \rightarrow \lambda_0$. Our results extend those by Stuart [21]. We use degree theory if the multiplicity of λ_0 is odd and Morse theory (or more specifically, Gromoll-Meyer theory) if it is not.

1. INTRODUCTION

In this paper we consider the Schrödinger equation

$$(1.1) \quad -\Delta u + V(x)u = \lambda u + f(x, u), \quad x \in \mathbb{R}^N,$$

where λ is a real parameter, $V \in L^\infty(\mathbb{R}^N)$, $f(x, u)/u \rightarrow m(x)$ as $|u| \rightarrow \infty$, $m \in L^\infty(\mathbb{R}^N)$ and λ_0 is an isolated eigenvalue of finite multiplicity for $\mathcal{L} := -\Delta + V(x) - m(x)$. \mathcal{L} will be considered as an operator in $L^2(\mathbb{R}^N)$. It is well known (see e.g. [18]) that \mathcal{L} is selfadjoint and its domain $D(\mathcal{L})$ is the Sobolev space $H^2(\mathbb{R}^N)$. We shall show that if the distance from λ_0 to the essential spectrum $\sigma_e(\mathcal{L})$ of \mathcal{L} is larger than the Lipschitz constant of $f - m$ (with respect to the u -variable), then there exists a sequence of solutions $(u_n, \lambda_n) \subset H^2(\mathbb{R}^N) \times \mathbb{R}$ such that $\|u_n\| \rightarrow \infty$ and $\lambda_n \rightarrow \lambda_0$. See Theorems 1.3 and 1.4 for more precise statements. We shall say that these solutions *bifurcate from infinity* or that λ_0 is an *asymptotic bifurcation point*. Our results extend those by Stuart [21] who has shown using degree theory that if $f(x, u) = f(u) + h(x)$, then asymptotic bifurcation occurs if λ_0 is of odd multiplicity and the bifurcating set contains a continuum.

Both here and in [21] (see also [20]) the result is first formulated in terms of an abstract operator equation. Let E be a Hilbert space, $L : D(L) \rightarrow E$ a selfadjoint linear operator and let $N : E \rightarrow E$ be a continuous nonlinear operator which is asymptotically linear in the sense of Hadamard (H -asymptotically linear for short, see Definition 2.1(i)). We show that if λ_0 is an isolated eigenvalue of odd multiplicity for L and if the distance $\text{dist}(\lambda_0, \sigma_e(L))$ from λ_0 to the essential spectrum of L is larger than the asymptotic Lipschitz constant of N (introduced in Definition 2.1(ii)), then λ_0 is an asymptotic bifurcation point for the equation

$$(1.2) \quad Lu = \lambda u + N(u), \quad u \in D(L).$$

Here we have assumed for notational simplicity that the asymptotic derivative $N'(\infty)$ of N is 0, see Theorem 1.1 for the full statement. This theorem slightly extends some results in [20, 21]

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where the distance condition on λ_0 was somewhat stronger. If N is the gradient of a C^1 -functional and λ_0 is an isolated eigenvalue of finite (not necessarily odd) multiplicity, we show that under an additional hypothesis λ_0 is an asymptotic bifurcation point for (1.2). The exact statement is given in Theorem 1.2. Existence of asymptotic bifurcation when the multiplicity of λ_0 is even seems to be new and is the main abstract result of this paper. A related problem $u = \lambda(Au + N(u))$ has been considered in [6, 23] under the assumptions that A is bounded linear, $A + N$ is the gradient of a functional and a k -set contraction, and N is asymptotically linear *in the stronger sense of Fréchet*. It was then shown that each eigenvalue $1/\lambda_0$ of A with $|\lambda_0 k| < 1$ is an asymptotic bifurcation point. However, the arguments there seem to break down in our case.

The proofs in [20, 21] were effected by first making the inversion $u \mapsto u/\|u\|^2$ (an idea that goes back to Rabinowitz [16] and Toland [22]). In this way the problem is transformed to that of looking for bifurcation from 0 instead of infinity. In the next step a finite-dimensional reduction is performed and finally it is shown that since λ_0 has odd multiplicity, the Brouwer degree for the linearization of the reduced operator at $u = 0$ changes as λ passes through λ_0 . This forces bifurcation, and an additional argument which goes back to [15] and uses degree theory in an essential way, shows that there is a continuum bifurcating from $(0, \lambda_0)$. Since the degree does not change if the multiplicity of λ_0 is even, in Theorem 1.2 we use Morse theory instead, and therefore we need the assumption that N is the gradient of a functional. Morse theory can only assert that there exists a sequence, and not necessarily a continuum, bifurcating from infinity. Let us also point out that in [20] a more general operator equation of the form $F(\lambda, u) = 0$ has been considered ($F(\lambda, \cdot)$ acts between two Banach spaces). Here we will only be concerned with (1.2), and this allows some simplifications of Stuart's arguments (in particular in the part involving the finite-dimensional reduction). Since we do not make inversion, we get a less restrictive bound for the distance from λ_0 to the essential spectrum.

The fact that $\text{dist}(\lambda_0, \sigma_e(L))$ is larger than the Lipschitz constant of N at infinity is needed in order to perform a finite-dimensional reduction of Liapunov-Schmidt type. As we shall see, if the distance condition is satisfied, then one can find an orthogonal decomposition $E = Z \oplus W$, where $\dim Z < \infty$, such that writing $u = z + w \in Z \oplus W$, it is possible to use the contraction mapping principle in order to express w as a function of z and λ . Although one may think this is only a technical condition, it has been shown by Stuart [21, Section 5.2] that there exist examples where asymptotic bifurcation does not occur at eigenvalues of odd multiplicity (and in Section 5.3 there one finds an example where asymptotic bifurcation occurs when λ_0 is not an eigenvalue). So the above condition, or some other, is needed.

The reason for requiring N to be H -asymptotically and not just asymptotically linear (in the sense of Fréchet) is that, in contrast to the situation when (1.1) is considered for x in a bounded domain, we cannot expect the Nemytskii operator N induced by f to be asymptotically linear. Indeed, it has been shown in [19] that if $f(u)/u \rightarrow m$ as $|u| \rightarrow \infty$, then N is always H -asymptotically linear, and it is asymptotically linear if and only if $f(u) = mu$. In the proof of Theorem 1.3 we show that also the Nemytskii operator corresponding to $f(x, u)$ is H -asymptotically linear if $f(x, u)/u \rightarrow m(x)$ as $|u| \rightarrow \infty$. The related concept of H -differentiability in the context of elliptic equations in \mathbb{R}^N has been introduced in a series of papers by Evéquoz and Stuart, see e.g. [7].

Now we can state our main results. The symbols $N'(\infty)$ and Lip_∞ (denoting asymptotic H -derivative and asymptotic Lipschitz constant) which appear below are introduced in Definition 2.1.

Theorem 1.1. *Let E be a Hilbert space and suppose that $L : D(L) \rightarrow E$ is a selfadjoint linear operator. Suppose further that*

- (i) N is H -asymptotically linear and $N'(\infty) : E \rightarrow E$ is selfadjoint,
- (ii) λ_0 is an isolated eigenvalue of odd multiplicity for $L - N'(\infty)$ and

$$\text{Lip}_\infty(N - N'(\infty)) < \text{dist}(\lambda_0, \sigma_e(L - N'(\infty))).$$

Then λ_0 is an asymptotic bifurcation point for equation (1.2). Moreover, there exists a continuum bifurcating from infinity at λ_0 .

By a continuum bifurcating from infinity at λ_0 we mean a closed connected set $\Gamma \subset E \times \mathbb{R}$ of solutions of (1.2) which contains a sequence (u_n, λ_n) such that $\|u_n\| \rightarrow \infty$, $\lambda_n \rightarrow \lambda_0$. This theorem should be compared with Theorem 4.2 and Corollary 4.3 in [21] (see also Theorem 6.3 in [20]) where the distance condition was somewhat stronger than in (ii) above. The main ingredient in the proof is a finite-dimensional reduction which roughly speaking goes as follows. Let W be an L -invariant subspace of E such that $\text{codim } W < \infty$ and $Z := W^\perp \subset D(L)$. Let $P : E \rightarrow W$ be the orthogonal projection and write $w = Pu$, $z = (I - P)u$. Then (1.2) is equivalent to the system

$$\begin{aligned} Lw - \lambda w &= PN(w + z), \\ Lz - \lambda z &= (I - P)N(w + z). \end{aligned}$$

Choosing an appropriate W , $\delta > 0$ small enough and $R > 0$ large enough, one can solve uniquely for w in the first equation provided $|\lambda - \lambda_0| \leq \delta$ and $\|z\| \geq R$. In this way we obtain $w = w(\lambda, z)$ which inserted in the second equation gives a (finite-dimensional) problem on $Z \setminus B_R(0)$. See Proposition 3.4 for more details. Now the proof of Theorem 1.1 is completed by a well-known argument using Brouwer's degree.

If N is a potential operator, then the reduced problem has variational structure. More precisely, suppose $N(u) = \nabla \psi(u)$ for some $\psi \in C^1(E, \mathbb{R})$ and let $\Phi_\lambda(u) := \frac{1}{2} \langle Lu - \lambda u, u \rangle - \psi(u)$. Then the functional φ_λ given by $\varphi_\lambda(z) = \Phi_\lambda(w(\lambda, z) + z)$ is of class C^1 and $z \in Z \setminus \overline{B}_R(0)$ is a critical point of φ_λ if and only if $u = w(\lambda, z) + z$ is a solution of (1.2), see Proposition 3.6. Recall that a functional φ is said to satisfy the Palais-Smale condition ((PS) for short) if each sequence (z_n) such that $\varphi(z_n)$ is bounded and $\varphi'(z_n) \rightarrow 0$ contains a convergent subsequence.

Theorem 1.2. *Let E be a Hilbert space and suppose that $L : D(L) \rightarrow E$ is a selfadjoint linear operator. Suppose further that*

- (i) N is a potential operator, i.e. there exists a functional $\psi \in C^1(E, \mathbb{R})$ such that $\nabla \psi(u) = N(u)$ for all $u \in E$,
- (ii) N is H -asymptotically linear and $N'(\infty) : E \rightarrow E$ is selfadjoint,
- (iii) λ_0 is an isolated eigenvalue of finite multiplicity for $L - N'(\infty)$ and

$$\text{Lip}_\infty(N - N'(\infty)) < \text{dist}(\lambda_0, \sigma_e(L - N'(\infty))).$$

If φ_{λ_0} satisfies (PS), then λ_0 is an asymptotic bifurcation point for equation (1.2).

Note that here we do not assume λ_0 is of odd multiplicity. In Theorem 1.4 below we shall give sufficient conditions for f in order that such λ_0 be an asymptotic bifurcation point for (1.1).

To formulate our results for equation (1.1) we introduce the following assumptions on f :

- (f₁) $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory condition, i.e., it is continuous in s for almost all $x \in \mathbb{R}^N$ and measurable in x for all $s \in \mathbb{R}$, and there exist $\alpha \in L^2(\mathbb{R}^N)$, $\beta \in \mathbb{R}^+$ such that $|f(x, s)| \leq \alpha(x) + \beta|s|$ for all $x \in \mathbb{R}^N$, $s \in \mathbb{R}$;
- (f₂) f is Lipschitz continuous in the second variable, with Lipschitz constant $\text{Lip}(f) := \inf\{C : |f(x, s) - f(x, t)| \leq C|s - t| \text{ for all } x \in \mathbb{R}^N, s, t \in \mathbb{R}\}$;
- (f₃) $\lim_{|s| \rightarrow \infty} f(x, s)/s = m(x)$, where $m \in L^\infty(\mathbb{R}^N)$;
- (f₄) $g(x, s) := f(x, s) - m(x)s$ is bounded by a constant independent of $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$;
- (f₅) Assume the limits $g_\pm(x) := \lim_{s \rightarrow \pm\infty} g(x, s)$ exist and either $\pm g_\pm \geq 0$ a.e. or $\pm g_\pm \leq 0$ a.e. In addition, there exists a set of positive measure on which none of g_\pm vanishes;
- (f₆) Assume the limits $h_\pm(x) := \lim_{s \rightarrow \pm\infty} g(x, s)s$ exist, $h_\pm \in L^\infty(\mathbb{R}^N)$ and either $g(x, s)s \geq 0$ or $g(x, s)s \leq 0$ for all $x \in \mathbb{R}^N$, $s \in \mathbb{R}$. In addition, there exists a set of positive measure on which none of h_\pm vanishes.

Note that if $f(x, s) = \alpha(x) + f_0(s)$ and $|f_0(s)| \leq \beta|s|$, where $\alpha \in L^2(\mathbb{R}^N)$, $\beta > 0$ and f_0 is continuous, then f satisfies (f₁). As we have already mentioned, such functions f have been considered in [21].

Theorem 1.3. *Suppose that $V \in L^\infty(\mathbb{R}^N)$ and f satisfies (f₁)-(f₃). Let $g(x, s) := f(x, s) - m(x)s$. If λ_0 is an isolated eigenvalue of odd multiplicity for $-\Delta + V - m$ and $\text{Lip}(g) < \text{dist}(\lambda_0, \sigma_e(-\Delta + V - m))$, then λ_0 is an asymptotic bifurcation point for equation (1.1). Moreover, there exists a continuum bifurcating from infinity at λ_0 .*

This strengthens some of the results of [21, Theorem 5.2]. Using examples in [21, Theorems 5.4, 5.6] and the remarks following them we shall show in Remark 5.1 that the condition on $\text{Lip}(g)$ above is sharp in the sense that if $\text{Lip}(g) > \text{dist}(\lambda_0, \sigma_e(-\Delta + V - m))$, then there may be no bifurcation at a simple eigenvalue.

Theorem 1.4. *Suppose that $V \in L^\infty(\mathbb{R}^N)$ and f satisfies (f₁)-(f₄) and either (f₅) or (f₆). If λ_0 is an isolated eigenvalue of finite multiplicity for $-\Delta + V - m$ and $\text{Lip}(g) < \text{dist}(\lambda_0, \sigma_e(-\Delta + V - m))$, then λ_0 is an asymptotic bifurcation point for equation (1.1).*

To our knowledge there are no earlier results on asymptotic bifurcation for (1.1) if the multiplicity of λ_0 is even.

The rest of the paper is organized as follows. Section 2 contains some preliminary material. In Section 3 a finite-dimensional reduction is performed. In Section 4 we prove Theorems 1.1 and 1.2, and Section 5 is concerned with the proofs of Theorems 1.3 and 1.4.

Notation. $\langle \cdot, \cdot \rangle$ denotes the inner product in a (real) Hilbert space E and $\|\cdot\|$ is the corresponding norm. If $\Phi \in C^1(E, \mathbb{R})$, then $\Phi'(u) \in E^*$ is the Fréchet derivative of Φ at u and $\nabla\Phi(u)$ (the gradient of Φ at u) is the corresponding element in E , i.e., $\langle \nabla\Phi(u), v \rangle = \Phi'(u)v$. The graph norm

corresponding to a linear operator L will be denoted by $\|\cdot\|_L$. The symbol $B_r(a)$ will stand for the open ball centered at a and having radius r , and we denote the L^p -norm of u by $\|u\|_p$.

2. PRELIMINARIES

Let X, Y be (real) Banach spaces and let $N : X \setminus B_R(0) \rightarrow Y$.

Definition 2.1. (i) We say that N is *asymptotically linear in the sense of Hadamard* (*H-asymptotically linear* for short) if there is a bounded linear operator $B : X \rightarrow Y$ such that

$$\lim_{n \rightarrow \infty} \frac{N(t_n u_n)}{t_n} = Bu$$

for all sequences $(t_n) \subset \mathbb{R}$, $(u_n) \subset X$ such that $u_n \rightarrow u$ and $\|t_n u_n\| \rightarrow \infty$. The operator B is called the *asymptotic H-derivative* and is denoted by $N'(\infty)$.

(ii) We say that N is *Lipschitz continuous at infinity* if

$$\text{Lip}_\infty(N) := \lim_{R \rightarrow \infty} \sup \left\{ \frac{\|N(u) - N(v)\|}{\|u - v\|} : u \neq v, \|u\|, \|v\| \geq R \right\} < \infty.$$

Note that the limit is well defined because the supremum above decreases as R increases.

Remark 2.2. (i) The definition of *H-asymptotic linearity* given in [19] is in fact a little different but the one formulated above is somewhat more convenient and is equivalent to the original one as has been shown in [19, Theorem A.1].

(ii) Recall that N is *asymptotically linear* (in the sense of Fréchet) if there is a bounded linear operator B such that

$$(2.1) \quad \lim_{\|u\| \rightarrow \infty} \frac{\|N(u) - Bu\|}{\|u\|} = 0.$$

It is clear that if N is asymptotically linear, then it is *H-asymptotically linear* and $N'(\infty) = B$. If, however, $\dim X < \infty$, then *H-asymptotic linearity* is equivalent to asymptotic linearity and (2.1) above holds for $B = N'(\infty)$, see [19, Remark 2].

Recall that a linear operator $L : D(L) \subset X \rightarrow Y$ is called a *Fredholm operator* if it is densely defined, closed, $\dim N(L) < \infty$ (where $N(L)$ is the kernel of L), the range $R(L)$ is closed and $\text{codim } R(L) < \infty$. The number

$$\text{ind}(L) := \dim N(L) - \text{codim } R(L)$$

is the *index of L* (cf. [17, Section 1.3]).

Suppose that E is a real Hilbert space and let $L : D(L) \subset E \rightarrow E$ be a selfadjoint Fredholm operator. Then $\text{ind}(L) = 0$, $E = N(L) \oplus R(L)$ (orthogonal sum) and $S := L|_{R(L) \cap D(L)}$ is invertible with bounded inverse. Hence, in view of [9, Problem III.6.16],

$$\|S^{-1}\| = r(S^{-1}) = \frac{1}{\text{dist}(0, \sigma(S))} = \frac{1}{\text{dist}(0, \sigma(L) \setminus \{0\})},$$

where $r(S^{-1})$ denotes the spectral radius of S^{-1} . The first equality holds since S^{-1} is selfadjoint, see [9, (V.2.4)]. Recall that a selfadjoint operator is necessarily densely defined and closed.

It is clear that if W is a closed subspace of $R(L)$, invariant with respect to L (i.e. $L(W \cap D(L)) \subset W$), then $L_W := L|_{W \cap D(L)}$ is also invertible and

$$\|L_W^{-1}\| = \frac{1}{\text{dist}(0, \sigma(L_W))}.$$

Remark 2.3. Keeping the above notation observe that $L_W^{-1} : W \rightarrow W \cap D(L)$ is bounded with respect to the graph norm $\|\cdot\|_L$ in $W \cap D(L)$ (recall that $\|u\|_L := \|u\| + \|Lu\|$ for $u \in D(L)$). In fact,

$$\|L_W^{-1}w\|_L = \|L_W^{-1}w\| + \|w\| \leq \left(1 + \frac{1}{\text{dist}(0, \sigma(L_W))}\right) \|w\|, \quad w \in W.$$

Definition 2.4. For a selfadjoint Fredholm operator $L : D(L) \rightarrow E$, let us put

$$(2.2) \quad \gamma(L) := \inf\{\|(L|_{W \cap D(L)})^{-1}\| : W \in \mathcal{W}\},$$

where \mathcal{W} denotes the family of closed L -invariant linear subspaces of $R(L)$ such that $\text{codim } W < \infty$ and $W^\perp \subset D(L)$.

Definition 2.5. By the *essential spectrum* $\sigma_e(L)$ of a selfadjoint linear operator $L : E \supset D(L) \rightarrow E$ we understand the set

$$\{\lambda \in \mathbb{C} : L - \lambda I \text{ is not a Fredholm operator}\}$$

(see [17, §1.4]).

It follows immediately from this definition that $\sigma_e(L) \subset \sigma(L)$ and $\sigma(L) \setminus \sigma_e(L)$ consists of isolated eigenvalues of finite multiplicity.

Theorem 2.6. *Let $L : E \supset D(L) \rightarrow E$ be a selfadjoint linear operator and let $\lambda_0 \in \sigma(L) \setminus \sigma_e(L)$. Then $L - \lambda_0 I$ is a Fredholm operator and*

$$\gamma(L - \lambda_0 I) = \frac{1}{\text{dist}(\lambda_0, \sigma_e(L))}.$$

If $\sigma_e(L) = \emptyset$ (this is the case e.g. if L is resolvent compact), then $\gamma(L - \lambda_0 I) = 0$.

Proof. Since $\sigma_e(L) - \lambda_0 = \sigma_e(L - \lambda_0 I)$ and hence

$$\text{dist}(\lambda_0, \sigma_e(L)) = \text{dist}(0, \sigma_e(L - \lambda_0 I)),$$

we may assume without loss of generality that $\lambda_0 = 0$ and we will show that

$$\gamma(L) = \frac{1}{\text{dist}(0, \sigma_e(L))}.$$

If $W \in \mathcal{W}$ and $Z := W^\perp$, then $\dim Z < \infty$ and $Z \subset D(L)$ is L -invariant. Hence $\sigma(L) = \sigma(L|_{W \cap D(L)}) \cup \sigma(L|_Z)$. Obviously, any $\lambda \in \sigma(L|_Z)$ is an isolated eigenvalue of finite multiplicity; thus $\sigma_e(L) \subset \sigma(L|_{W \cap D(L)})$. This implies that

$$\|(L|_{W \cap D(L)})^{-1}\| = \frac{1}{\text{dist}(0, \sigma(L|_{W \cap D(L)}))} \geq \frac{1}{\text{dist}(0, \sigma_e(L))} \text{ and therefore } \gamma(L) \geq \frac{1}{\text{dist}(0, \sigma_e(L))}.$$

Take any $0 < d < \text{dist}(0, \sigma_e(L))$ and let

$$D = [-d, d] \cap \sigma(L), \quad B := \sigma(L) \setminus D.$$

Clearly D is finite: if $\lambda \in D$, then $\lambda \in \sigma(L) \setminus \sigma_e(L)$, i.e., λ is an isolated eigenvalue of finite multiplicity. Therefore B is closed and $\sigma_e(L) \subset B$. Obviously, $\sigma(L) = D \cup B$. Let Z be the subspace spanned by the eigenfunctions corresponding to the eigenvalues in D and let $W = Z^\perp$. Then $Z \subset D(L)$, $W \subset R(L)$, Z, W are invariant with respect to L , $L|_Z$ is bounded, $D = \sigma(L|_Z)$ and $B = \sigma(L|_{W \cap D(L)})$. Clearly, $W \in \mathcal{W}$ since $\dim Z < \infty$. Now

$$\|(L|_{W \cap D(L)})^{-1}\| = r((L|_{W \cap D(L)})^{-1}) = \frac{1}{\text{dist}(0, \sigma(L|_{W \cap D(L)}))} = \frac{1}{\text{dist}(0, B)} \leq \frac{1}{d}.$$

This implies the assertion. Note that if $\sigma_e(L) = \emptyset$, we can choose any $d > 0$. Hence $\gamma(L) = 0$. \square

Remark 2.7. Let L be a Fredholm operator of index 0 and let $\mathcal{P}(L)$ denote the collection of all bounded operators K of finite rank and such that $L + K$ is invertible. Clearly, $\mathcal{P}(L) \neq \emptyset$. Put

$$\tilde{\gamma}(L) := \inf\{\|(L + K)^{-1}\| : K \in \mathcal{P}(L)\}.$$

Then $\tilde{\gamma}(L)$ corresponds to the notion of *essential conditioning number* in [20, Section 5.1], see also [21, Section 3.1] where the definition above appears explicitly.

We claim that if L is a selfadjoint Fredholm operator, then $\tilde{\gamma}(L) = \gamma(L)$. For $K \in \mathcal{P}(L)$, $\sigma_e(L) = \sigma_e(L + K) \subset \sigma(L + K)$, hence

$$\|(L + K)^{-1}\| \geq r((L + K)^{-1}) = \frac{1}{\text{dist}(0, \sigma(L + K))} \geq \frac{1}{\text{dist}(0, \sigma_e(L))}.$$

So $\tilde{\gamma}(L) \geq \gamma(L)$ according to the definition of $\tilde{\gamma}$ and Theorem 2.6. On the other hand, take any $W \in \mathcal{W}$ and let $Z := W^\perp$. As before, write $u = z + w \in Z \oplus W$ and let $Ku := \alpha z - Lz$, where

$$\alpha := \inf\{\|Lw\| : w \in W \cap D(L), \|w\| = 1\} > 0.$$

Then K has finite rank and, for $u \in D(L)$, $Lu + Ku = Lw + \alpha z$. Hence $L + K$ is invertible and it is easy to see that

$$\inf\{\|Lu + Ku\| : u \in D(L), \|u\| = 1\} \geq \alpha.$$

So

$$\tilde{\gamma}(L) \leq \|(L + K)^{-1}\| \leq \frac{1}{\alpha} = \|(L|_{W \cap D(L)})^{-1}\|$$

and $\tilde{\gamma}(L) \leq \gamma(L)$. We have shown that $\tilde{\gamma}(L) = \gamma(L)$. Therefore Theorem 2.6 may be considered as a refinement of [20, Theorem 5.5 and Corollary 5.6].

3. THE PROBLEM AND FINITE-DIMENSIONAL REDUCTION

Let E be a real Hilbert space and $L : E \supset D(L) \rightarrow E$ a selfadjoint operator. We shall study the existence of solutions to the eigenvalue problem (1.2), i.e.,

$$Lu = \lambda u + N(u), \quad u \in D(L), \quad \lambda \in \mathbb{R},$$

or, more precisely, the existence of asymptotic bifurcation of solutions to (1.2). Recall that $\lambda_0 \in \mathbb{R}$ is an *asymptotic bifurcation point* for (1.2) if there exist sequences $\lambda_n \rightarrow \lambda_0$ and $(u_n) \subset D(L)$ such that $\|u_n\| \rightarrow \infty$ and $Lu_n - N(u_n) = \lambda_n u_n$.

By X we denote the domain $D(L)$ furnished with the graph norm

$$\|u\|_L := \|u\| + \|Lu\|, \quad u \in D(L).$$

Then X is a Banach space, L is bounded as an operator from X to E and the inclusion $i : X \hookrightarrow E$ is continuous.

If N is a potential operator, i.e. there exists $\psi \in C^1(E, \mathbb{R})$ such that $N = \nabla \psi$, then along with (1.2) we can consider the existence of critical points of the functional $\Phi_\lambda : X \rightarrow \mathbb{R}$, $\lambda \in \mathbb{R}$, given by

$$\Phi_\lambda(u) := \frac{1}{2} \langle Lu - \lambda u, u \rangle - \psi(u), \quad u \in X.$$

Since $|\langle Lu, u \rangle| \leq \|Lu\| \|u\| \leq \|u\|_L^2$, $\Phi_\lambda \in C^1(X, \mathbb{R})$ and

$$(3.1) \quad \Phi'_\lambda(u)v = \langle Lu - \lambda u, v \rangle - \langle N(u), v \rangle, \quad u, v \in X.$$

It is clear that if $u \in X$ solves (1.2) for some $\lambda \in \mathbb{R}$, then $\Phi'_\lambda(u)v = 0$ for all $v \in X$, i.e., u is a critical point of Φ_λ . Conversely, if $u \in X$ and $\Phi'_\lambda(u) = 0$, then u solves (1.2) since $D(L)$ is dense in E . Note that if L is unbounded, then Φ_λ is defined on $D(L)$ and is not C^1 with respect to the original norm $\|\cdot\|$ of E on $D(L)$.

In what follows we assume:

3.1. N is H -asymptotically linear with $N'(\infty) = 0$;

3.2. N is Lipschitz continuous at infinity;

3.3. $\lambda_0 = 0 \in \sigma(L) \setminus \sigma_e(L)$ and $\text{Lip}_\infty(N) < \text{dist}(0, \sigma_e(L))$.

Observe that these assumptions cause no loss of generality in Theorems 1.1 and 1.2 since if $N'(\infty) \neq 0$ is selfadjoint and $\lambda_0 \neq 0$, then we may replace L by $L - N'(\infty) - \lambda_0 I$ and N by $N - N'(\infty)$.

As a first step towards showing that $\lambda_0 = 0$ is an asymptotic bifurcation point for (1.2) we perform a kind of a Liapunov-Schmidt finite-dimensional reduction near infinity. Put

$$L_\lambda u := Lu - \lambda u, \text{ where } u \in D(L_\lambda) = D(L), \lambda \in \mathbb{R}$$

and note that the norms $\|\cdot\|_L$ and $\|\cdot\|_{L_\lambda}$ are equivalent. Given $W \in \mathcal{W}$, let $P : E \rightarrow W$ be the orthogonal projection and $Z := W^\perp$. Observe that $u = w + z \in D(L)$, where $w \in W$, $z \in Z$, solves (1.2) if and only if

$$(3.2) \quad L_\lambda w = PN(w + z),$$

$$(3.3) \quad L_\lambda z = (I - P)N(w + z).$$

Proposition 3.4. *There are a subspace $W \in \mathcal{W}$, numbers $\delta \in (0, \text{dist}(0, \sigma(L) \setminus \{0\}))$, $R > 0$ and a continuous map $w : [-\delta, \delta] \times (Z \setminus B_R(0)) \rightarrow W \cap D(L)$ such that (3.2) holds for $w = w(\lambda, z)$ and:*

(i) *For any λ with $|\lambda| \leq \delta$, $z, z' \in Z \setminus B_R(0)$ and some constant $c > 0$,*

$$(3.4) \quad \|w(\lambda, z) - w(\lambda, z')\| \leq \|w(\lambda, z) - w(\lambda, z')\|_L \leq c\|z - z'\|.$$

In particular, $w(\cdot, \cdot)$ is continuous with respect to the graph norm.

(ii) *$w(\lambda, \cdot)$ is H -asymptotically linear with $w'(\lambda, \infty) = 0$.*

(iii) *$z \in Z \setminus B_R(0)$ is a solution of (3.3) with $w = w(\lambda, z)$ if and only if $u = w(\lambda, z) + z$ is a solution of (1.2).*

Note that the condition on δ implies invertibility of L_λ for $0 < |\lambda| \leq \delta$.

Proof. (i) According to Definition 2.4 of $\gamma(L)$, Theorem 2.6 and assumption 3.3, there is a closed subspace $W \in \mathcal{W}$ for which

$$\text{Lip}_\infty(N) \|(L|_{W \cap D(L)})^{-1}\| < 1.$$

Hence we can find $\delta \in (0, \text{dist}(0, \sigma(L) \setminus \{0\}))$ and $R > 0$ such that

$$k := \sup_{|\lambda| \leq \delta} \|(L_\lambda|_{W \cap D(L)})^{-1}\| \cdot \beta < 1,$$

where

$$(3.5) \quad \beta := \sup \left\{ \frac{\|N(u) - N(v)\|}{\|u - v\|} : u \neq v, \|u\|, \|v\| \geq R \right\}.$$

Let $Z := W^\perp$ and let $P : E \rightarrow W$ be the orthogonal projection. To facilitate the notation let us put

$$M_\lambda(w + z) := (L_\lambda|_{W \cap D(L)})^{-1} P N(w + z) \in W \cap D(L), \quad w \in W, z \in Z \text{ and } |\lambda| \leq \delta.$$

Then (3.2) is equivalent to the fixed point equation

$$(3.6) \quad w = M_\lambda(w + z).$$

Fix $\lambda \in [-\delta, \delta]$ and $z \in Z$, $\|z\| \geq R$. If $w, w' \in W$, then $\|w + z\|, \|w' + z\| \geq \|z\| \geq R$, so taking into account that $\|P\| = 1$, we have

$$\|M_\lambda(w + z) - M_\lambda(w' + z)\| \leq k\|w - w'\|.$$

By the Banach contraction principle there is a unique $w = w(\lambda, z) \in W \cap D(L)$, continuously depending on λ and z , such that (3.6), and hence (3.2), holds. Moreover,

$$\begin{aligned} \|w(\lambda, z) - w(\lambda, z')\| &= \|M_\lambda(w(\lambda, z) + z) - M_\lambda(w(\lambda, z') + z')\| \leq \\ &k\|w(\lambda, z) - w(\lambda, z') + z - z'\| \leq k\|w(\lambda, z) - w(\lambda, z')\| + k\|z - z'\| \end{aligned}$$

for all $|\lambda| \leq \delta$, $z, z' \in Z \setminus B_R(0)$. So $\|w(\lambda, z) - w(\lambda, z')\| \leq k(1 - k)^{-1}\|z - z'\|$. Using this, (3.5) and arguing as above, we obtain

$$\begin{aligned} \|L_\lambda w(\lambda, z) - L_\lambda w(\lambda, z')\| &= \|PN(w(\lambda, z) + z) - PN(w(\lambda, z') + z')\| \\ &\leq \beta\|w(\lambda, z) - w(\lambda, z')\| + \beta\|z - z'\| \leq \frac{\beta}{1 - k}\|z - z'\|. \end{aligned}$$

Since $\|\cdot\|_L$ and $\|\cdot\|_{L_\lambda}$ are equivalent norms, the second inequality in (3.4) follows (the first one is obvious).

(ii) To show the H -asymptotic linearity of $w(\lambda, \cdot)$ with $w'(\lambda, \infty) = 0$, let $(z_n) \subset Z$ and $(t_n) \subset \mathbb{R}$ be sequences such that $z_n \rightarrow z$ and $\|t_n z_n\| \rightarrow \infty$. Then, for sufficiently large n , $\|w(\lambda, t_n z_n) + t_n z_n\| \geq \|t_n z_n\| \geq R$ and

$$\|w(\lambda, t_n z_n)\| \leq \|M_\lambda(w(\lambda, t_n z_n) + t_n z_n) - M_\lambda(t_n z_n)\| + \|M_\lambda(t_n z_n)\| \leq k\|w(\lambda, t_n z_n)\| + \|M_\lambda(t_n z_n)\|.$$

Thus, in view of assumption 3.1,

$$(3.7) \quad \frac{\|w(\lambda, t_n z_n)\|}{|t_n|} \leq \frac{1}{1 - k} \frac{\|M_\lambda(t_n z_n)\|}{|t_n|} \rightarrow 0.$$

(iii) is an immediate consequence of (i). \square

Remark 3.5. Suppose that $z_n \rightarrow z$ in Z and take a sequence $(t_n) \subset \mathbb{R}$ such that $\|t_n z_n\| \rightarrow \infty$. Then, again in view of the H -asymptotic linearity of N and (3.7), we have

$$(3.8) \quad \frac{N(w(\lambda, t_n z_n) + t_n z_n)}{t_n} = \frac{N\left(t_n \left(\frac{w(\lambda, t_n z_n)}{t_n} + z_n\right)\right)}{t_n} \rightarrow 0$$

for each fixed $\lambda \in [-\delta, \delta]$.

If $N = \nabla\psi$, then we let

$$(3.9) \quad \varphi_\lambda(z) := \Phi_\lambda(w(\lambda, z) + z), \quad |\lambda| \leq \delta, \quad z \in Z \setminus \overline{B}_R(0).$$

Proposition 3.6. *Let $|\lambda| \leq \delta$. Then $\varphi_\lambda \in C^1(Z \setminus \overline{B}_R(0), \mathbb{R})$ and*

$$(3.10) \quad \nabla\varphi_\lambda(z) = L_\lambda z - (I - P)N(w(\lambda, z) + z).$$

Therefore $z \in Z \setminus \overline{B}_R(0)$ is a critical point of φ_λ if and only if $u = w(\lambda, z) + z$ solves (1.2). Moreover, $\nabla\varphi_\lambda$ is asymptotically linear with $(\nabla\varphi_\lambda)'(\infty) = L_\lambda|_Z$.

Proof. To show (3.10) we shall compute the derivative of φ_λ in the direction $h \in Z$, $h \neq 0$. For notational convenience we write $w(z)$ for $w(\lambda, z)$. Let $t > 0$,

$$u := w(z) + z \quad \text{and} \quad \xi := w(z + th) - w(z) + th.$$

Then we have

$$\varphi_\lambda(z + th) - \varphi_\lambda(z) = \Phi_\lambda(u + \xi) - \Phi_\lambda(u) - \Phi'_\lambda(u)\xi + \Phi'_\lambda(u)\xi.$$

Clearly, $\xi \neq 0$ as $t > 0$. In view of (3.1), (3.2) and since $w(z + th) - w(z) \in W$,

$$\begin{aligned} \Phi'_\lambda(u)\xi &= \langle L_\lambda u - N(u), \xi \rangle = \langle L_\lambda w(z) - PN(u), \xi \rangle + \langle L_\lambda z - (I - P)N(u), \xi \rangle \\ &= \langle L_\lambda z - N(u), th \rangle = t\Phi'_\lambda(u)h. \end{aligned}$$

Hence

$$(3.11) \quad \frac{\varphi_\lambda(z + th) - \varphi_\lambda(z)}{t} = \Phi'_\lambda(u)h + \frac{\|\xi\|_L}{t} \cdot \frac{\Phi_\lambda(u + \xi) - \Phi_\lambda(u) - \Phi'_\lambda(u)\xi}{\|\xi\|_L}.$$

It follows from (3.4) that

$$\|\xi\|_L \leq td\|h\|$$

for some $d > 0$. This, together with the Fréchet differentiability of Φ_λ on X (i.e., on $D(L)$ with the graph norm) implies that the second term on the right-hand side of (3.11) tends to 0 as $t \rightarrow 0$. So

$$\lim_{t \rightarrow 0^+} \frac{\varphi_\lambda(z + th) - \varphi_\lambda(z)}{t} = \Phi'_\lambda(u)h = \langle L_\lambda z, h \rangle - \langle (I - P)N(w(z) + z), h \rangle.$$

Therefore φ_λ is continuously Gâteaux differentiable, hence continuously Fréchet differentiable as well, and the derivative is as claimed.

If $z \in Z \setminus \overline{B}_R(0)$ is a critical point of φ_λ , then (3.3) with $w = w(\lambda, z)$ is satisfied; this together with (3.2) shows that $u = w(\lambda, z) + z$ solves (1.2).

Since $\dim Z < \infty$, in order to prove the last part of the assertion it suffices to show that $\nabla\varphi_\lambda$ is H -asymptotically linear (see Remark 2.2(ii)). If $z_n \rightarrow z$ in Z , $(t_n) \subset \mathbb{R}$ and $\|t_n z_n\| \rightarrow \infty$, then, in view of (3.8),

$$\frac{\nabla\varphi_\lambda(t_n z_n)}{t_n} = L_\lambda z_n - \frac{(I - P)N(w(t_n z_n) + t_n z_n)}{t_n} \rightarrow L_\lambda z.$$

This concludes the proof. \square

Remark 3.7. (i) Using (3.4) and the fact that β in (3.5) is finite, it is easy to see that $\nabla\varphi_\lambda$ is Lipschitz continuous on $Z \setminus \overline{B}_R(0)$ and the Lipschitz constant may be chosen independently of $\lambda \in [-\delta, \delta]$.

(ii) In what follows we may (and will need to) assume that φ_λ is defined on Z and not only on $Z \setminus \overline{B}_R(0)$. Such an extension of φ_λ can be achieved e.g. as follows. Let $\chi \in C^\infty(\mathbb{R}, [0, 1])$ be a cutoff function such that $\chi(t) = 0$ for $t \leq R + 1$ and $\chi(t) = 1$ for $t \geq R + 2$. Set $\tilde{\varphi}_\lambda(z) := \chi(\|z\|)\varphi_\lambda(z)$. Then $\tilde{\varphi}_\lambda$ is of class C^1 , Lipschitz continuous and $\tilde{\varphi}_\lambda(z) = \varphi_\lambda(z)$ for $\|z\| > R + 2$. In particular, $z \in Z \setminus \overline{B}_{\tilde{R}}(0)$, where $\tilde{R} := R + 2$, is a critical point of $\tilde{\varphi}_\lambda$ if and only if $u = w(\lambda, z) + z$ solves (1.2).

4. PROOFS OF THEOREMS 1.1 AND 1.2

In the proof of Theorem 1.1 we shall need the following version of Whyburn's lemma which may be found in [1, Proposition 5]:

Lemma 4.1. *Let Y be a compact space and $A, B \subset Y$ closed sets. If there is no connected set $\Gamma \subset Y \setminus (A \cup B)$ such that $\overline{\Gamma} \cap A \neq \emptyset$ and $\overline{\Gamma} \cap B \neq \emptyset$ ($\overline{\Gamma}$ stands for the closure of Γ in Y), then A and B are separated, i.e. there are open sets $U, V \subset Y$ such that $A \subset U$, $B \subset V$, $U \cap V = \emptyset$ and $Y = U \cup V$ (clearly, U, V are closed as well).*

Proof of Theorem 1.1. By Proposition 3.4, it suffices to consider equation (3.3) with $w = w(\lambda, z)$ which we re-write in the form

$$(4.1) \quad F_\lambda(z) := L_\lambda z - (I - P)N(w(\lambda, z) + z) = 0.$$

As in assumptions 3.1–3.3, it causes no loss of generality to take $\lambda_0 = 0$ and $N'(\infty) = 0$. Although F_λ in Proposition 3.4 has been defined for $|\lambda| \leq \delta$ and $\|z\| \geq R$, we may (and do) extend it continuously to $[-\delta, \delta] \times Z$. Since $w'(\lambda, \infty) = 0$ (see (ii) of Proposition 3.4) and asymptotic linearity coincides with H -asymptotic linearity on Z (because $\dim Z < \infty$), we have, setting $K_\lambda(z) := (I - P)N(w(\lambda, z) + z)$ and using Remark 3.5,

$$(4.2) \quad \lim_{\|z\| \rightarrow \infty} \frac{\|K_\lambda(z)\|}{\|z\|} = 0.$$

Suppose there is no asymptotic bifurcation at $\lambda_0 = 0$. Taking smaller δ and larger R if necessary, $F_\lambda(z) \neq 0$ for any $|\lambda| \leq \delta$ and $\|z\| \geq R$. Therefore the Brouwer degree $\deg(F_\lambda, B_R(0), 0)$ (see e.g. [2, Section 3.1]) is well defined and independent of $\lambda \in [-\delta, \delta]$. Since $\delta < \text{dist}(0, \sigma(L) \setminus \{0\})$, $L_{\pm\delta}$ are invertible. It follows therefore from (4.2) that if $R_0 \geq R$ is sufficiently large, then $L_{\pm\delta}z - tK_{\pm\delta}(z) \neq$

0 for any $\|z\| \geq R_0$, $t \in [0, 1]$. Hence by the excision property and the homotopy invariance of degree,

$$k = \deg(F_{\pm\delta}, B_R(0), 0) = \deg(F_{\pm\delta}, B_{R_0}(0), 0) = \deg(L_{\pm\delta}|_Z, B_{R_0}, 0)$$

for some $k \in \mathbb{Z}$. Let d_1, d_2 be the number of negative eigenvalues (counted with their multiplicity) of respectively $L_\delta|_Z$ and $L_{-\delta}|_Z$. Then $k = (-1)^{d_1} = (-1)^{d_2}$ [2, Lemma 3.3]. However, since $d_1 = d_2 + \dim N(L)$ and $\dim N(L)$ is odd, this is impossible. So we have reached a contradiction to the assumption that there is no bifurcation.

It remains to prove that there exists a bifurcating continuum. Usually this is done by first making the inversion $u \mapsto u/\|u\|^2$ and then showing there is a continuum bifurcating from 0 [16, 20, 22]. Here we give a slightly different argument avoiding inversion. Let

$$\Sigma := \{(z, \lambda) \in (Z \setminus B_R(0)) \times [-\delta, \delta] : F_\lambda(z) = 0\}.$$

Compactify Z by adding the point at infinity and let $A := \overline{B}_R(0) \times [-\delta, \delta]$, $B := \{(\infty, 0)\}$, $Y := A \cup \Sigma \cup B$. Then Y is compact, A and B are closed disjoint. We claim that if R is large enough, there is a connected set $\Gamma \subset \Sigma$ such that $\{(\infty, 0)\} \in \overline{\Gamma}$ (the closure taken in Y) and $\overline{\Gamma} \cap A \neq \emptyset$. Otherwise there exist U and V as in Lemma 4.1. Since U is compact and bounded, there exists a bounded open set $\mathcal{O} \subset Z \times [-\delta, \delta]$ such that $U \subset \mathcal{O}$ and $\partial\mathcal{O} \cap \Sigma = \emptyset$. Letting $\mathcal{O}_\lambda := \{z : (z, \lambda) \in \mathcal{O}\}$ for $\lambda \in [-\delta, \delta]$, it follows from the excision property and the generalized version of the homotopy invariance property of degree [2, Theorem 4.1] that $\deg(F_\delta, \mathcal{O}_\delta, 0) = \deg(F_{-\delta}, \mathcal{O}_{-\delta}, 0)$, a contradiction since by the same argument as above $\deg(F_\delta, \mathcal{O}_\delta, 0) = (-1)^{k_1}$, $\deg(F_{-\delta}, \mathcal{O}_{-\delta}, 0) = (-1)^{k_2}$ and k_1, k_2 have different parity. \square

In the proof of Theorem 1.2 we shall use Gromoll-Meyer theory. Below we summarize some pertinent facts which are special cases of much more general results of [12] where functionals were considered in a Hilbert space E with filtration, i.e., with a sequence (E_n) of subspaces such that $E_n \subset E_{n+1}$ for all n and $\bigcup_{n=1}^\infty E_n$ is dense in E . In the terminology of [12], here we have the trivial filtration (i.e., $Z_n = Z$ for all n) which, together with the fact that $\dim Z < \infty$, considerably simplifies the proofs. An alternative approach is via the Conley index theory, see e.g. [3, 4], in particular [3, Corollary 2.3] and [4, Theorem 2].

Let $\varphi : Z \rightarrow \mathbb{R}$ be a function such that $\nabla\varphi$ is locally Lipschitz continuous. Suppose also $K = K(\varphi) := \{z \in Z : \nabla\varphi(z) = 0\}$ is bounded. A pair $(\mathbb{W}, \mathbb{W}^-)$ of closed subsets of Z will be called *admissible* (for φ and K) if

- (i) $K \subset \text{int}(\mathbb{W})$ and $\mathbb{W}^- \subset \partial\mathbb{W}$;
- (ii) $\varphi|_{\mathbb{W}}$ is bounded;
- (iii) There exist a locally Lipschitz continuous vector field V defined in a neighbourhood N of \mathbb{W} and a continuous function $\beta : N \rightarrow \mathbb{R}^+$ such that $\|V(z)\| \leq 1$, $\langle V(z), \varphi(z) \rangle \geq \beta(z)$ for all $z \in N$, and β is bounded away from 0 on compact subsets of $N \setminus K$ (we shall call V *admissible* for $(\mathbb{W}, \mathbb{W}^-)$);
- (iv) \mathbb{W}^- is a piecewise C^1 -manifold of codimension 1, V is transversal to \mathbb{W}^- , the flow η of $-V$ can leave \mathbb{W} only via \mathbb{W}^- and if $z \in \mathbb{W}^-$, then $\eta(t, z) \notin \mathbb{W}$ for any $t > 0$.

Let H^* denote the Čech (or Alexander-Spanier) cohomology with coefficients in \mathbb{Z}_2 and let the critical groups $c^*(\varphi, K)$ of the pair (φ, K) be defined by

$$c^*(\varphi, K) := H^*(\mathbb{W}, \mathbb{W}^-).$$

Lemma 4.2. *Suppose φ satisfies (PS).*

(i) *For each $R > 0$ there exists a bounded admissible pair $(\mathbb{W}, \mathbb{W}^-)$ for φ and K such that $B_R(0) \subset \mathbb{W}$.*

(ii) *If $(\mathbb{W}_1, \mathbb{W}_1^-)$ and $(\mathbb{W}_2, \mathbb{W}_2^-)$ are two admissible pairs for φ and K , then $H^*(\mathbb{W}_1, \mathbb{W}_1^-) \cong H^*(\mathbb{W}_2, \mathbb{W}_2^-)$ (i.e., $c^*(\varphi, K)$ is well defined).*

(iii) *Suppose $\{\varphi_\lambda\}_{\lambda \in [0,1]}$ is a family of functions satisfying (PS) and such that $\nabla \varphi_\lambda$ is locally Lipschitz continuous, $\lambda \mapsto \nabla \varphi_\lambda$ is continuous, uniformly on bounded subsets of Z , and $K(\varphi_\lambda) \subset B_R(0)$ for some $R > 0$ and all $\lambda \in [0, 1]$. Then $c^*(\varphi_\lambda, K(\varphi_\lambda))$ is independent of λ .*

This lemma corresponds to Lemma 2.13 and Propositions 2.12, 2.14 in [12]. Note that condition (PS)* there is in our setting (i.e. for trivial filtration) equivalent to (PS).

Outline of proof. (i) Choose R, a, b so that $K \subset B_R(0)$, $a < \varphi(z) < b$ for all $z \in B_R(0)$ and let

$$(4.3) \quad V(z) := \frac{\nabla \varphi(z)}{1 + \|\nabla \varphi(z)\|}.$$

Clearly, the flow η given by

$$\frac{d\eta}{dt} = -V(\eta), \quad \eta(0, z) = z$$

is defined on $\mathbb{R} \times Z$. Let

$$\mathbb{W} := \{\eta(t, z) : t \geq 0, z \in B_R(0), \varphi(\eta(t, z)) \geq a\}, \quad \mathbb{W}^- := \mathbb{W} \cap \varphi^{-1}(a).$$

Then $(\mathbb{W}, \mathbb{W}^-)$ is an admissible pair. The proof follows that of [12, Lemma 2.13] but is simpler - there is no need for using cutoff functions. Note that (here and below) the Palais-Smale condition rules out the possibility that $\varphi(\eta(t, z)) > a$ and $\|\eta(t, z)\| \rightarrow \infty$ as $t \rightarrow \infty$, hence $t \mapsto \eta(t, z)$ either approaches K as $t \rightarrow \infty$ or hits $\mathbb{W}^- = \varphi^{-1}(a)$ in finite time.

(ii) Assume that φ is unbounded below and above (the other cases are simpler but somewhat different). Let $(\mathbb{W}_0, \mathbb{W}_0^-)$ be an admissible pair and V_0 a corresponding admissible vector field. As $\varphi|_{\mathbb{W}_0}$ is bounded, we may choose a, b so that $a < \varphi(z) < b$ for all $z \in \mathbb{W}_0$. Since $(\mathbb{W}_1, \mathbb{W}_1^-) := (\varphi^{-1}([a, b]), \varphi^{-1}(a))$ is an admissible pair, it suffices to show that $H^*(\mathbb{W}_0, \mathbb{W}_0^-) \cong H^*(\mathbb{W}_1, \mathbb{W}_1^-)$. Put $V(z) := \chi_0(z)V_0(z) + \chi_1(z)V_1(z)$, where V_1 is given by (4.3) and $\{\chi_0, \chi_1\}$ is a Lipschitz continuous partition of unity such that $\chi_0(z) = 1$ on \mathbb{W}_0 and $\chi_1(z) = 1$ in a neighbourhood of $\partial \mathbb{W}_1$. Denote the flow of $-V$ by η . Let $A := \{\eta(t, z) : t \geq 0, z \in \mathbb{W}_0^-\} \cap \mathbb{W}_1$ and $\mathbb{W} = \mathbb{W}_0 \cup A$, $\mathbb{W}^- := \mathbb{W} \cap \mathbb{W}_1^-$. Then $(\mathbb{W}, \mathbb{W}^-)$ is an admissible pair and using η one obtains a strong deformation retraction of A onto \mathbb{W}^- . So $H^*(A, \mathbb{W}^-) = 0$ and by exactness of the cohomology sequence of the triple $(\mathbb{W}, A, \mathbb{W}^-)$ and the strong excision property we have $H^*(\mathbb{W}, \mathbb{W}^-) \cong H^*(\mathbb{W}, A) \cong H^*(\mathbb{W}_0, \mathbb{W}_0^-)$. We also have, by excision again, $H^*(\mathbb{W}, \mathbb{W}^-) \cong H^*(\mathbb{W} \cup \mathbb{W}_1^-, \mathbb{W}_1^-)$. Finally, using the flow η once more, we obtain a deformation of $(\mathbb{W}_1, \mathbb{W}_1^-)$ into $(\mathbb{W} \cup \mathbb{W}_1^-, \mathbb{W}_1^-)$ which leaves $\mathbb{W} \cup \mathbb{W}_1^-$ and \mathbb{W}_1^- invariant. Hence $(\mathbb{W} \cup \mathbb{W}_1^-, \mathbb{W}_1^-)$ and $(\mathbb{W}_1, \mathbb{W}_1^-)$ are homotopy equivalent and thus have the same

cohomology. Putting everything together gives $H^*(\mathbb{W}_0, \mathbb{W}_0^-) \cong H^*(\mathbb{W}_1, \mathbb{W}_1^-)$. More details of the proof may be found in [12, Propositions 2.12 and 2.7].

(iii) Let $\lambda_0 \in [0, 1]$. It suffices to show that $c^*(\varphi_\lambda, K(\varphi_\lambda))$ is constant for λ in a neighbourhood of λ_0 . Denote the vector field for φ_λ given as in (4.3) by V_λ and choose an admissible pair $(\mathbb{W}_{\lambda_0}, \mathbb{W}_{\lambda_0}^-)$ for φ_{λ_0} and $K(\varphi_{\lambda_0})$ such that $B_{R_1}(0) \subset \mathbb{W}_{\lambda_0}$, where $R_1 > R$. By the construction in (i), we may assume V_{λ_0} is admissible for this pair. Let $\tilde{V}(z) := \chi_1(z)V_\lambda(z) + \chi_2(z)V_{\lambda_0}(z)$, where $\{\chi_1, \chi_2\}$ is a partition of unity subordinate to the sets $B_{R_1}(0)$ and $\mathbb{W}_{\lambda_0} \setminus \overline{B}_R(0)$. It is easy to see that if $|\lambda - \lambda_0|$ is small enough, then $(\mathbb{W}_{\lambda_0}, \mathbb{W}_{\lambda_0}^-)$ is an admissible pair for φ_λ , $K(\varphi_\lambda)$ and \tilde{V} is a corresponding admissible field. Note in particular that

$$\|\nabla\varphi_\lambda(z)\| \geq \|\nabla\varphi_{\lambda_0}(z)\| - \|\nabla\varphi_\lambda(z) - \nabla\varphi_{\lambda_0}(z)\| > 0$$

for $z \in \mathbb{W}_{\lambda_0} \setminus \overline{B}_R(0)$, so indeed \tilde{V} is admissible. Hence $c^*(\varphi_\lambda, K(\varphi_\lambda)) \cong c^*(\varphi_{\lambda_0}, K(\varphi_{\lambda_0}))$. \square

Proof of Theorem 1.2. Let φ_λ be given by (3.9) and extend it to the whole space Z according to Remark 3.7. If $\lambda_0 = 0$ is not an asymptotic bifurcation point for (1.2), then it follows from Proposition 3.6 that $\nabla\varphi_\lambda(z) \neq 0$ for $\lambda \in [-\delta, \delta]$ and $\|z\| > R$, possibly after choosing a smaller δ and larger R . By assumption, φ_0 satisfies (PS) and since L_λ has bounded inverse if $0 < |\lambda| \leq \delta$, we see using (4.2) that $\nabla\varphi_\lambda$ is bounded away from 0 as $\|z\| \rightarrow \infty$. Hence all φ_λ , $|\lambda| \leq \delta$, satisfy (PS). By Lemma 4.2, $c^*(\varphi_\lambda, K(\varphi_\lambda))$ is independent of $\lambda \in [-\delta, \delta]$. For $\lambda = \delta$, let $Z = Z_\delta^+ \oplus Z_\delta^-$ and $z = z^+ + z^- \in Z_\delta^+ \oplus Z_\delta^-$, where Z_δ^\pm are the maximal L_δ -invariant subspaces of Z on which L_δ is respectively positive and negative definite. Choose $\varepsilon > 0$ such that $\langle \pm L_\delta z^\pm, z^\pm \rangle \geq \varepsilon \|z^\pm\|^2$ and let

$$\mathbb{W} := \{z \in Z : \|z^+\| \leq R_0, \|z^-\| \leq R_0\}, \quad \mathbb{W}^- := \{z \in \mathbb{W} : \|z^-\| = R_0\}.$$

Recall $K_\lambda(z) = (I - P)N(w(\lambda, z) + z)$. Taking a sufficiently large R_0 ,

$$\langle \nabla\varphi_\delta(z), z^+ \rangle = \langle L_\delta z, z^+ \rangle - \langle K_\delta(z), z^+ \rangle \geq \varepsilon \|z^+\|^2 - \frac{1}{4}\varepsilon \|z\| \|z^+\| > 0, \quad z \in \mathbb{W}, \|z^+\| = R_0.$$

Similarly,

$$\langle \nabla\varphi_\delta(z), z^- \rangle < 0, \quad z \in \mathbb{W}, z^- \in \mathbb{W}^-.$$

So the flow of $-\nabla\varphi_\delta$ is transversal to \mathbb{W}^- and can leave \mathbb{W} only via \mathbb{W}^- . Hence $(\mathbb{W}, \mathbb{W}^-)$ is an admissible pair for φ_δ and $K(\varphi_\delta)$, and $V = \nabla\varphi_\delta$ is a corresponding admissible vector field. Note that this pair is also admissible for the quadratic functional $\Psi_\delta(z) := \frac{1}{2}\langle L_\delta z, z \rangle$. Since 0 is the only critical point of Ψ_δ , it follows e.g. from [14, Corollary 8.3] that if m is the Morse index of Ψ_δ , then

$$c^q(\varphi_\delta, K(\varphi_\delta)) = c^q(\Psi_\delta, 0) = \delta_{q,m}\mathbb{Z}_2.$$

A similar argument shows that $c^q(\varphi_{-\delta}, K(\varphi_{-\delta})) = \delta_{q,n}\mathbb{Z}_2$, where n is the Morse index of $\Psi_{-\delta}$. As the Morse index changes (by $\dim N(L)$) when λ passes through 0, $m \neq n$ and $c^*(\varphi_\delta, K(\varphi_\delta)) \neq c^*(\varphi_{-\delta}, K(\varphi_{-\delta}))$. This is the desired contradiction. \square

5. PROOFS OF THEOREMS 1.3 AND 1.4

We assume throughout this section that $V \in L^\infty(\mathbb{R}^N)$ and f satisfies (f_1) – (f_3) . We consider equation (1.1) which we re-write in the form

$$(5.1) \quad -\Delta u + V_0(x)u = \lambda u + g(x, u), \quad x \in \mathbb{R}^N,$$

where we have put $V_0(x) := V(x) - m(x)$ and $g(x, u) := f(x, u) - m(x)u$. Let λ_0 be an isolated eigenvalue of finite multiplicity for $-\Delta + V_0$. Replacing $V_0(x)$ by $V_0(x) - \lambda_0$ we may assume without loss of generality that $\lambda_0 = 0$.

Let $E := L^2(\mathbb{R}^N)$ and $Lu := -\Delta u + V_0(x)u$. As we have pointed out in the introduction, L is a selfadjoint operator whose domain is the Sobolev space $H^2(\mathbb{R}^N)$ and the graph norm of L is equivalent to the Sobolev norm. (A brief argument: using the Fourier transform one readily sees that $-\Delta + 1 : H^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is an isomorphism; hence the conclusion follows because $V \in L^\infty(\mathbb{R}^N)$.)

We define the operator N (the Nemytskii operator) by setting

$$N(u) := g(\cdot, u(\cdot)), \quad u \in E.$$

It follows from (f_1) and Krasnoselskii's theorem [11, Theorems 2.1 and 2.3] that $N : E \rightarrow E$ is well defined and continuous. Let

$$G(x, s) := \int_0^s g(x, \xi) d\xi, \quad x \in \mathbb{R}^N, \quad s \in \mathbb{R}$$

and

$$\psi(u) := \int_{\mathbb{R}^N} G(x, u) dx, \quad u \in E.$$

Then $\psi \in C^1(E, \mathbb{R})$ and

$$\nabla \psi(u) = N(u),$$

see [11, Lemma 5.1]. Furthermore, let

$$\Phi_\lambda(u) := \frac{1}{2} \langle Lu - \lambda u, u \rangle - \psi(u), \quad u \in X := H^2(\mathbb{R}^N).$$

Then $\Phi_\lambda \in C^1(X, \mathbb{R})$ and $\Phi'_\lambda(u) = 0$ if and only if u is a solution of (5.1).

Proof of Theorem 1.3. We verify the assumptions of Theorem 1.1. First we show that N is H -asymptotically linear and $N'(\infty) = 0$. Let $u_n \rightarrow u$ and $\|t_n u_n\| \rightarrow \infty$ in E . Assume passing to a subsequence that $u_n \rightarrow u$ a.e. Since

$$\frac{g(x, t_n u_n)^2}{t_n^2} \leq \left(\frac{\alpha(x)}{t_n} + (\beta + \|m\|_\infty) |u_n| \right)^2$$

and $g(x, s)/s \rightarrow 0$ as $|s| \rightarrow \infty$, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{n \rightarrow \infty} \frac{\|N(u_n)\|^2}{t_n^2} = \int_{\mathbb{R}^N} \lim_{n \rightarrow \infty} \frac{g(x, t_n u_n)^2}{(t_n u_n)^2} u_n^2 dx = 0.$$

Hence (i) of Theorem 1.1 is satisfied. Since $\text{Lip}_\infty(N) = \text{Lip}_\infty(g) \leq \text{Lip}(g) < \text{dist}(0, \sigma_e(L))$ (where the second inequality follows by assumption), also (ii) of this theorem holds. This completes the proof. \square

Remark 5.1. As we have mentioned in the introduction, the condition $\text{Lip}(g) < \text{dist}(\lambda_0, \sigma_e(L))$ is sharp in the sense that there may be no asymptotic bifurcation if $\text{Lip}(g) > \text{dist}(\lambda_0, \sigma_e(L))$ and other assumptions of Theorem 1.3 are satisfied. Let $N = 1$ and suppose $V_0 \in C^1(\mathbb{R})$, $V_0'(x) \leq 0$ for x large, $\lim_{|x| \rightarrow \infty} V_0(x) = V_0(\infty)$ exists and $\inf\{\langle Lu, u \rangle : \|u\|_2 = 1\} < V_0(\infty)$. Then $\sigma_e(L) = [V_0(\infty), \infty)$ and $\lambda_0 := \inf \sigma(L) < \inf \sigma_e(L)$ is a simple eigenvalue. Assume without loss of generality that $\lambda_0 = 0$. Assume also that g is independent of x , of class C^1 , $g(0) = \lim_{|s| \rightarrow \infty} g(s)/s = 0$ and $\xi := V_0(\infty) + g'(0) < 0$. Given $\varepsilon > 0$, we may choose g so that $\text{Lip}(g) = -g'(0) \in (V_0(\infty), V_0(\infty) + \varepsilon)$. So

$$\text{Lip}(g) - \varepsilon < \text{dist}(0, \sigma_e(L)) = V_0(\infty) < \text{Lip}(g),$$

and according to [21, Theorem 5.4] and the remarks following it, there is no asymptotic bifurcation at any $\lambda > \xi$, in particular, not at $\lambda_0 = 0$. See also the explicit Example 1 after the proof of Theorem 5.4 in [21]. A similar conclusion holds for $N \geq 2$, see [21, Theorem 5.6].

In the proof of Theorem 1.4 we shall need an auxiliary result. Let $\lambda_0 = 0$ and write $w(z) = w(0, z)$. Then $w(z)$ satisfies equation (3.2), i.e. we have

$$Lw(z) = PN(w(z) + z).$$

Lemma 5.2. *Suppose (f_1) – (f_4) are satisfied. Then $\|w(z)\|_\infty \leq C$ for some constant $C > 0$ and all $\|z\| > R$.*

Proof. Recall $L := -\Delta + V_0$, where $L : D(L) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$. We also define $\tilde{L} := -\Delta + V_0$ when $-\Delta + V_0$ is regarded as an operator in $L^\infty(\mathbb{R}^N)$ (i.e., $\tilde{L} : D(\tilde{L}) \subset L^\infty(\mathbb{R}^N) \rightarrow L^\infty(\mathbb{R}^N)$). By [8, Theorem], $\sigma(L) = \sigma(\tilde{L})$ and isolated eigenvalues of L and \tilde{L} have the same multiplicity. Since Z is spanned by eigenfunctions of $-\Delta + V_0$ corresponding to isolated eigenvalues and such eigenfunctions decay exponentially [18, Theorem C.3.4], $Z \subset L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. It follows therefore from [9, Theorem III.6.17] that there is an L -invariant decomposition $L^\infty(\mathbb{R}^N) = \tilde{Z} \oplus \tilde{W}$, where $\tilde{Z} = Z$. Moreover, by [9, (III.6.19)],

$$Q := I - P = -\frac{1}{2\pi i} \int_\gamma (L - \lambda I)^{-1} d\lambda,$$

where γ is a smooth simple closed curve (in \mathbb{C}) which encloses all eigenvalues corresponding to Z and no other points in $\sigma(L)$. By [8, Proposition 2.1], $(L - \lambda I)^{-1}|_{L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)} = (\tilde{L} - \lambda I)^{-1}|_{L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)}$. Hence $Q|_{L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)} = \tilde{Q}|_{L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)}$, where \tilde{Q} denotes the \tilde{L} -invariant projection of $L^\infty(\mathbb{R}^N)$ on Z , and the same equality holds for P and $\tilde{P} := I - \tilde{Q}$. \tilde{P} is a projection on a subspace of finite codimension, hence it is continuous and therefore (f_1) , (f_4) imply $y = y(z) := PN(w(z) + z) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\|y\|_\infty \leq C_1$ for some C_1 independent of $z \in Z \setminus \overline{B}_R(0)$. Since $L|_{\tilde{W}}$ has bounded inverse, $\|\tilde{w}\|_\infty \leq C$, where $\tilde{w} = \tilde{w}(z) := \tilde{L}^{-1}y$ (note that for $w = w(z) = L^{-1}y$ we only have a z -dependent L^2 -bound because $N(w + z)$ is not uniformly bounded in $L^2(\mathbb{R}^N)$).

We complete the proof by showing that $w = \tilde{w}$. Let $\mu_n \notin \sigma(L)$, $\mu_n \rightarrow 0$. By the resolvent equation [9, (I.5.5) and §III.6.1],

$$w = L^{-1}y = (L - \mu_n I)^{-1}y - \mu_n L^{-1}(L - \mu_n I)^{-1}y$$

and

$$\tilde{w} = \tilde{L}^{-1}y = (\tilde{L} - \mu_n I)^{-1}y - \mu_n \tilde{L}^{-1}(\tilde{L} - \mu_n I)^{-1}y.$$

Let $v_n := (L - \mu_n I)^{-1}y$. Since $y \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, [8, Proposition 2.1] implies $v_n = (\tilde{L} - \mu_n I)^{-1}y$ as well. As the last term on each of the right-hand sides above tends to 0, (v_n) is a Cauchy sequence in $L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ (with the norm $\|\cdot\|_{L^2 \cap L^\infty} := \|\cdot\|_2 + \|\cdot\|_\infty$) which yields $w = \tilde{w}$. \square

Proof of Theorem 1.4. We have already shown that assumptions (i)-(iii) of Theorem 1.2 are satisfied. Suppose first that (f_4) and (f_5) hold. We only need to verify that φ_0 satisfies (PS). Recall from (3.9) that for $\|z\| > R$

$$\varphi_0(z) = \Phi_0(w(z) + z),$$

where we have put $w(z) = w(0, z)$, and by Proposition 3.6, we have

$$(5.2) \quad \langle \nabla \varphi_0(z), \zeta \rangle = \langle Lz, \zeta \rangle - \int_{\mathbb{R}^N} g(x, w(z) + z) \zeta \, dx \quad \text{for all } z, \zeta \in Z, \, \|z\| > R.$$

Let $z = z^+ + z^- + z^0 \in Z^+ \oplus Z^- \oplus Z^0$, where Z^+, Z^- respectively denote the subspaces of Z corresponding to the positive and the negative part of the spectrum of $L|_Z$ and $Z^0 := N(L) \subset Z$. Let $(z_n) \subset Z$ be such that $\nabla \varphi_0(z_n) \rightarrow 0$. It suffices to consider z_n with $\|z_n\| > R$, and we shall show that (z_n) is bounded. Since Z is spanned by eigenfunctions of $-\Delta + V_0$ and $\dim Z < \infty$, it follows from [18, Theorem C.3.4] that there are constants $\delta, C_0 > 0$ such that $|z(x)| \leq C_0 e^{-\delta|x|}$ for all $x \in \mathbb{R}^N$ and all $z \in Z$ with $\|z\| \leq 1$. In particular, such z are uniformly bounded in $L^p(\mathbb{R}^N)$ for any $p \in [1, \infty]$. Using this, (f_4) and equivalence of norms in Z , we obtain

$$|\langle Lz_n^+, z \rangle| \leq \left| \int_{\mathbb{R}^N} g(x, w(z_n) + z_n) z \, dx \right| + o(1)\|z\| \leq c_1\|z\| \leq c_2 \quad \text{for all } z \in Z^+, \, \|z\| = 1.$$

Hence (z_n^+) is bounded and a similar argument shows that so is (z_n^-) . Suppose $\|z_n^0\| \rightarrow \infty$ and write $z_n^0 = t_n \tilde{z}_n^0$, where $\|\tilde{z}_n^0\| = 1$. Passing to a subsequence, $\tilde{z}_n^0 \rightarrow \tilde{z}^0 \in Z^0$. Denote

$$v_n := w(z_n) + z_n^+ + z_n^-.$$

We shall obtain a contradiction with the assumption $\nabla \varphi_0(z_n) \rightarrow 0$ by showing that

$$(5.3) \quad \langle -\nabla \varphi_0(z_n), \tilde{z}_n^0 \rangle = \int_{\mathbb{R}^N} g(x, v_n + t_n \tilde{z}_n^0) \tilde{z}_n^0 \, dx \not\rightarrow 0.$$

By Lemma 5.2, the sequence $(w(z_n))$ is bounded in $L^\infty(\mathbb{R}^N)$, and since so are the sequences (z_n^\pm) , $v_n(x) + t_n \tilde{z}_n^0(x) \rightarrow \pm\infty$ for all $x \in A_\pm := \{x \in \mathbb{R}^N : \pm \tilde{z}^0(x) > 0\}$.

Suppose $\pm g_\pm \geq 0$. Since g is bounded and \tilde{z}_n^0 is uniformly bounded in $L^1(\mathbb{R}^N)$, we may use the Lebesgue dominated convergence theorem to obtain

$$\lim_{n \rightarrow \infty} \int_{A_\pm} g(x, v_n + t_n \tilde{z}_n^0) \tilde{z}_n^0 \, dx = \int_{A_\pm} g_\pm \tilde{z}^0 \, dx \geq 0.$$

By the unique continuation property [5, Proposition 3 and Remark 2], $\tilde{z}^0(x) \neq 0$ a.e. Hence the measure of $\mathbb{R}^N \setminus (A_+ \cup A_-)$ is 0 and thus

$$(5.4) \quad \int_{A_+} g_+ \tilde{z}^0 \, dx + \int_{A_-} g_- \tilde{z}^0 \, dx > 0.$$

This implies (5.3). If $\pm g_{\pm} \leq 0$, the same argument remains valid after making some obvious changes.

Suppose now that (f_4) and (f_6) are satisfied. Here we do not know whether (PS) holds for φ_0 , however, we will construct an admissible pair directly by adapting an argument in [10], see in particular the proof of Theorem 4.5 there. Suppose $g(x, s)s \geq 0$ in (f_6) and let

$$\mathbb{W} := \{z \in Z : \|z^{\pm}\| \leq R_0, \|z^0\| \leq R_1\}, \quad \mathbb{W}^- := \{z \in \mathbb{W} : \|z^-\| = R_0 \text{ or } \|z^0\| = R_1\}$$

(R_0, R_1 to be determined). Boundedness of g and equivalence of norms in Z yield

$$\left| \int_{\mathbb{R}^N} g(x, w(z) + z) z^+ dx \right| \leq c_3 \|z^+\|.$$

Since $\langle \pm Lz, z^{\pm} \rangle \geq \varepsilon \|z^{\pm}\|^2$ for some $\varepsilon > 0$, $\langle \nabla \varphi_0(z), z^+ \rangle \geq \varepsilon \|z^+\|^2 - c_3 \|z^+\| > 0$ if $\|z^+\| = R_0$ and $\langle \nabla \varphi_0(z), z^- \rangle < 0$ if $\|z^-\| = R_0$ provided R_0 is large enough. We want to show that there exists a (large) R_1 such that $\langle \nabla \varphi_0(z), z^0 \rangle < 0$ for z with $\|z^-\| = R_0$ and $\|z^0\| = R_1$. Assuming the contrary, $\liminf_{n \rightarrow \infty} \langle \nabla \varphi_0(z_n), z_n^0 \rangle \geq 0$ for a sequence (z_n) such that $\|z_n^0\| \rightarrow \infty$. Below we use the same notation as in (5.3). We have

$$0 = \langle -\nabla \varphi_0(z_n), w(z_n) \rangle = \int_{\mathbb{R}^N} g(x, v_n + t_n \tilde{z}_n^0) w(z_n) dx,$$

$g(x, s) \rightarrow 0$ as $|s| \rightarrow \infty$ (because $h_{\pm} \in L^{\infty}(\mathbb{R}^N)$ by (f_6)) and $|g(x, v_n + t_n \tilde{z}_n^0) z_n^{\pm}| \leq c_4 e^{-\delta|x|}$. So according to the Lebesgue dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} g(x, v_n + t_n \tilde{z}_n^0) v_n dx = 0.$$

Hence Fatou's lemma and (f_6) give

$$\liminf_{n \rightarrow \infty} \int_{A_{\pm}} g(x, v_n + t_n \tilde{z}_n^0) t_n \tilde{z}_n^0 dx = \liminf_{n \rightarrow \infty} \int_{A_{\pm}} g(x, v_n + t_n \tilde{z}_n^0) (v_n + t_n \tilde{z}_n^0) dx \geq \int_{A_{\pm}} h_{\pm} dx \geq 0.$$

Since by assumption at least one of the integrals on the right-hand side is positive (possibly infinite),

$$\liminf_{n \rightarrow \infty} \langle -\nabla \varphi_0(z_n), z_n^0 \rangle = \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} g(x, v_n + t_n \tilde{z}_n^0) t_n \tilde{z}_n^0 dx > 0,$$

a contradiction. So R_1 exists as required and $(\mathbb{W}, \mathbb{W}^-)$ is an admissible pair. Now it is easy to see as in the proof of (iii) of Lemma 4.2 that this is also an admissible pair for $\varphi_{\pm\delta}$ if δ is small enough. As in the proof of Theorem 1.2 one shows that the critical groups for φ_{δ} and $\varphi_{-\delta}$ are different, and this forces bifurcation.

If $g(x, s)s \leq 0$, a similar argument shows that $\langle \nabla \varphi_0(z), z^0 \rangle > 0$ for some R_1 , hence the exit set for the flow is $\mathbb{W}^- := \{z \in \mathbb{W} : \|z^-\| = R_0\}$. \square

Remark 5.3. Note that (5.4) is a variant of the Landesman-Lazer condition introduced in [13] and Theorem 1.4 remains valid if one assumes (5.4) holds for all $z \in N(L)$. This is slightly less restrictive than (f_5) . The reason that we have chosen (f_5) is that it is a general condition on f , with no reference to eigenfunctions corresponding to λ_0 . (f_6) is a kind of strong resonance condition because $g(x, s) \rightarrow 0$ as $|s| \rightarrow \infty$. Note also that our arguments show that under the assumptions of Theorem 1.4 there is a uniform bound for solutions of (1.1) with $\lambda = \lambda_0$.

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REFERENCES

- [1] J. C. Alexander, *A primer on connectivity*, in: E. Fadell, G. Fournier eds., *Fixed Point Theory*, Lecture Notes in Math. vol. 886, Springer 1981, pp. 455–483.
- [2] A. Ambrosetti and A. Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge University Press 2007.
- [3] V. Benci, *A new approach to the Morse-Conley theory and some applications*, Ann. Mat. Pura Appl. (4) 158 (1991), 231–305.
- [4] E. N. Dancer, *Degenerate critical points, homotopy indices and Morse inequalities*, J. Reine Angew. Math. 350 (1984), 1–22.
- [5] D. G. de Figueiredo and J.-P. Gossez, *Strict monotonicity of eigenvalues and unique continuation*, Comm. PDE 17 (1992), 339–346.
- [6] J.-P. Dias and J. Hernández, *A remark on a paper by J. F. Toland and some applications to unilateral problems*, Proc. Roy. Soc. Edinburgh 75A, (1975/76), 179–182.
- [7] G. Evéquoz and C. A. Stuart, *Hadamard differentiability and bifurcation*, Proc. Roy. Soc. Edinburgh 137A (2007), 1249–1285.
- [8] R. Hempel and J. Voigt, *The spectrum of a Schrödinger operator in $L_p(\mathbb{R}^\nu)$ is p -independent*, Comm. Math. Phys. 104 (1986), 243–250.
- [9] T. Kato, *Perturbation Theory for Linear Operators*, Springer 1995.
- [10] P. Kokocki, *Connecting orbits for nonlinear differential equations at resonance*, J. Diff. Eq. 255 (2013), 1554–1575.
- [11] M. A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press 1964.
- [12] W. Kryszewski and A. Szulkin, *An infinite dimensional Morse theory with applications*, Trans. Amer. Math. Soc. 349 (1997), 3181–3234.
- [13] E. M. Landesman and A.C. Lazer, *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech. 19 (1970), 609–623.
- [14] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer 1989.
- [15] P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Func. Anal. 7 (1971), 487–513.
- [16] P. H. Rabinowitz, *On bifurcation from infinity*, J. Diff. Eq. 14 (1973), 462–475.
- [17] M. Schechter, *Spectra of Partial Differential Operators*, North-Holland 1971.
- [18] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. 7 (1982), 447–526.
- [19] C. A. Stuart, *Asymptotic linearity and Hadamard differentiability*, Nonlinear Analysis 75 (2012), 4699–4710.
- [20] C. A. Stuart *Bifurcation at isolated singular points of the Hadamard derivative*, Proc. Royal Soc. Edinburgh 144A (2014), 1027–1065.
- [21] C. A. Stuart, *Asymptotic bifurcation and second order elliptic equations on \mathbb{R}^N* , Ann. IHP - Analyse Non Linéaire (2014), <http://dx.doi.org/10.1016/j.anihpc.2014.09.003>.
- [22] J. F. Toland, *Asymptotic linearity and nonlinear eigenvalue problems*, Quart. J. Math. Oxford 24 (1973), 241–250.
- [23] J. F. Toland, *Bifurcation and asymptotic bifurcation for non-compact non-symmetric gradient operators*, Proc. Roy. Soc. Edinburgh 73A (1974/75), 137–147.

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